flow readouts can be provided that account for variations of all measured parameters.

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Accounting for Effects of a System Rotation on the Pressure-Strain Correlation

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Introduction

TURBULENT shear layers are known to be very sensitive to the imposition of a system rotation. The classic experiments of Johnston et al.,[1] for example, conducted in a channel rotated about a spanwise axis, show that the turbulence is augmented on the pressure side and diminished on the suction side, relative to the nonrotating flow. At high rates of rotation, turbulence is extinguished altogether on the suction side, and the flow is seen to assume a laminar-like state. Effects such as these have highlighted serious shortcomings in eddy-viscosity-based closures as these, in most cases, contain no explicit dependence on rotation.[2] Such a dependence is present in the exact equations for the Reynolds stresses, and thus closures based on the solution of either the algebraic[3] or the differential[4] forms of these equations can be relied on to reproduce, qualitatively at any rate, the correct response to a system rotation. In this Note, we consider certain aspects of modeling the effects of a system rotation on the Reynolds-stress equations and put forward proposals for extending the applicability of existing pressure-strain models to rotating coordinate systems. We check the proposals by computing a fully developed $\{U_i = [U(y),\ 0,\ 0]\}$ turbulent flow inside a long, straight channel subjected to rotation about a spanwise axis, i.e., $\Omega_i = (0,\ 0,\ \Omega)$. The channel has an infinite aspect ratio, and so no secondary flows are generated. Comparisons are made with the experimental data of Johnston et al.[1] and with the recent direct numerical simulations of Kristoffersen and Andersson.[5] Interest in this flow stems from its being numerically trivial to represent yet one that is representative of many practically relevant flows whose behavior is strongly dominated by a system rotation.[6,7] This study extends the validation to higher rotation rates than hitherto reported in the literature and, in doing so, obtains a somewhat unexpected result.

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Present Proposal

The equation for the Reynolds stresses in a rotating frame of reference are

$$\underbrace{U_k \frac{\partial \overline{u_i u_j}}{\partial x_k}}_{\text{Convection: } C_{ij}} = -\overbrace{\left(\overline{u_i u_k}\frac{\partial U_j}{\partial x_k} + \overline{u_j u_k}\frac{\partial U_i}{\partial x_k}\right)}^{\text{Shear production: } P_{ij}} \overbrace{- 2\Omega_k\left(\overline{u_j u_m}\epsilon_{ikm} + \overline{u_i u_m}\epsilon_{jkm}\right)}^{\text{Rotation production: } G_{ij}}$$
$$\overbrace{-\frac{\partial}{\partial x_k}\left[\overline{u_i u_j u_k} + \frac{1}{\rho}\left(\overline{p' u_i}\delta_{jk} + \overline{p' u_j}\delta_{ik}\right) - \nu\frac{\partial \overline{u_i u_j}}{\partial x_k}\right]}^{\text{Diffusion: } D_{ij}}$$
$$\underbrace{-2\nu\left(\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}}\right)}_{\text{Dissipation: } \epsilon_{ij}} + \underbrace{\overline{\frac{p'}{\rho}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}}_{\text{Redistribution: } \Phi_{ij}} \qquad (1)$$

We focus here on modeling the fluctuating pressure-strain correlations term Φ_{ij} in the preceding equation. The explicit appearance of the fluctuating pressure can be eliminated by taking the divergence for the fluctuating velocity u_i, thus obtaining a Poisson equation for p'. With the assumption of homogeneous turbulence, the solution to this equation can be expressed as

$$\Phi_{ij} \equiv \overline{\frac{p'}{\rho}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]} = -\frac{1}{4\pi}\int \overline{\frac{\partial^2 (u_k u_l)'}{\partial x_k' x_l'}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]}\frac{\text{dvol}}{\boldsymbol{r}}$$
$$-\frac{1}{2\pi}\int \frac{\partial U_k'}{\partial x_l'}\overline{\frac{\partial u_l'}{\partial x_k'}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]}\frac{\text{dvol}}{\boldsymbol{r}}$$
$$-\frac{1}{2\pi}\epsilon_{ijk}\Omega_j\int \overline{\frac{\partial u_k'}{\partial x_l'}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]}\frac{\text{dvol}}{\boldsymbol{r}} \qquad (2)$$

It is immediately clear that the effects of rotation must be accounted for explicitly in the models for Φ_{ij}. These models are fairly well established for stationary flows, and the question that arises here is how best to extend these to flows with a system rotation. In addressing this question, note that, in a rotating frame, both P_{ij} and G_{ij} (quantities that commonly appear in the models for Φ_{ij}) are dependent on the frame's rotation rate, whereas turbulence is, of course, entirely independent of the choice of coordinates system used to analyze the flow. A primary requirement in the model, therefore, is that it should yield identical results irrespective of whether it is used in conjunction with a fixed or rotating frame of reference. In this connection, it is helpful to discard the traditional, term-by-term approach to modeling Φ_{ij} in favor of a more widely based formulation that recognizes that a complete model for Φ_{ij} may be obtained as a combination of terms involving the turbulence anisotropy b_{ij}, the mean rate of strain tensor S_{ij}, and the mean vorticity tensor W_{ij}, viz.,

$$\Phi_{ij} = -\left(C_1\epsilon + C_1^* P_k\right)b_{ij} + C_2\epsilon\left(b_{ik}b_{kj} - \tfrac{1}{3}b_{kl}b_{kl}\delta_{ij}\right)$$
$$+ \left(C_3 - C_3^* II_b^{\frac{1}{2}}\right)kS_{ij} + C_4 k\left(b_{ik}S_{jk} + b_{jk}S_{ik} - \tfrac{2}{3}b_{kl}S_{kl}\delta_{ij}\right)$$
$$+ C_5 k\left(b_{ik}W_{jk} + b_{jk}W_{ik}\right) \qquad (3)$$

where

$$S_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right), \qquad W_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right)$$
$$b_{ij} = \frac{\overline{u_i u_j}}{\overline{u_q u_q}} - \frac{1}{3}\delta_{ij}, \qquad II_b = b_{ij}b_{ij} \qquad (4)$$

Thus the linear model of Launder et al.[8] (hereafter LRR), its simplification (LRR-IP), and the quadratic model of Speziale et al.[9]

Table 1 Coefficients of the LRR and SSG pressure-strain models

Model	C_1	C_1^*	C_2	C_3	C_3^*	C_4	C_5
LRR	3.0	0	0	0.8	0	1.75	1.31
LRR-IP	3.6	0	0	0.8	0	1.20	1.20
SSG	3.4	1.8	4.2	0.8	1.3	1.25	0.4

(hereafter SSG) can now be recovered simply by assigning appropriate values to the various coefficients (Table 1).

All of the pressure-strain models that can be represented by Eq. (3) can now be extended to a rotating frame of reference by recognizing that the rate of strain tensor is a frame-indifferent tensor (and is thus independent of Ω_i) but that the vorticity tensor depends on the motion of the frame of reference. Thus, to obtain a coordinate-independent formulation, the vorticity tensor that appears in Eq. (3) should be replaced by the intrinsic spin tensor W_{ij}^*, i.e., the vorticity tensor relative to an inertial frame of reference.[10,11] This is defined as

$$W_{kl}^* = W_{kl} + \epsilon_{mkl}\Omega_m \qquad (5)$$

The preceding is a generalization of existing pressure-strain models to flows subjected to all modes of rotation. Examples of practically relevant flows that can thus be analyzed would therefore include those in noncircular ducts rotated about their longitudinal axis[6] and those in circular pipes rotated about an external axis.[7]

Results and Discussion

We check the rotating-frame formulation of both the LRR-IP and SSG models. The LRR-IP model is used in conjunction with standard wall reflection corrections. These are recognized to be troublesome in complex geometries, and often, as was the case here, their influence does not diminish rapidly with distance from the wall. In contrast, the SSG model, being properly calibrated for homogeneous shear flow,[12] is used without such corrections, its performance in simple wall-bounded flows being indistinguishable from that of the LRR models. Closure of Eq. (1) is completed by neglecting the pressure diffusion term and by adopting a simple gradient-transport model for the turbulence triple correlations. Further, isotropic dissipation is assumed, and the scalar turbulence energy dissipation rate is obtained from the solution of the standard equation. In obtaining the predictions that follow, convection was dropped from all equations, whereas cross-stream diffusion was evaluated by second-order-accurate central differencing. The streamwise momentum equation reduces to a balance between turbulent transport and the reduced pressure gradient, which is constant across the channel and is calculated, iteratively, from global mass-flow considerations. The computational grid consisted of 90 nodes unevenly distributed across the channel. The results are indistinguishable from those obtained with a grid of 50 nodes.

Comparisons are made with the experimental data of Johnston et al.[1] for a Reynolds number ($\equiv U_{\text{bulk}}H/\nu$) of 11.5×10^4 and a rotation number N ($\equiv \Omega H/U_{\text{bulk}}$) of 0.068 and 0.21. The predicted and measured cross-stream distributions of streamwise velocity are compared in Fig. 1. Also plotted there are Laufer's measurements in a stationary flow at $Re = 5.2 \times 10^4$. The effects of rotation are such that the turbulence activity is enhanced on the (destabilized) pressure side—and diminished on the (stabilized) suction side—relative to the stationary channel. The outcome, as can be seen from Fig. 1, is a highly nonsymmetric velocity distribution that is predicted fairly well by both models, for small N at any rate.

The predicted and measured values of the wall friction velocity (normalized by the zero rotation value) on the pressure and suction sides are presented in Fig. 2. Also plotted there are the low-Reynolds-number direct numerical simulation (DNS) results.[5] The effects of rotation on the friction velocity are what might at first be expected. On the destabilized pressure side, the wall shear stress initially rises and then appears to reach a saturated level at $N \approx 0.08$. On the stabilized suction side, the wall shear stress falls rapidly below the stationary-channel levels, and then at $N \approx 0.08$ in the experiments, it appears to collapse in a manner suggesting the occurrence of reverse transition. There is little agreement between the measurements and the DNS results, even after taking into consideration the differences in the respective Reynolds numbers. Turbulence closures, such as the present, that utilize the wall function approach cannot predict the rapid collapse nor, incidentally, is this behavior better reproduced by low-Reynolds-number versions of these models (e.g., Ref. 13).

Figure 2 reveals a somewhat surprising behavior for the wall shear stress on the pressure side. This quantity, after initially increasing

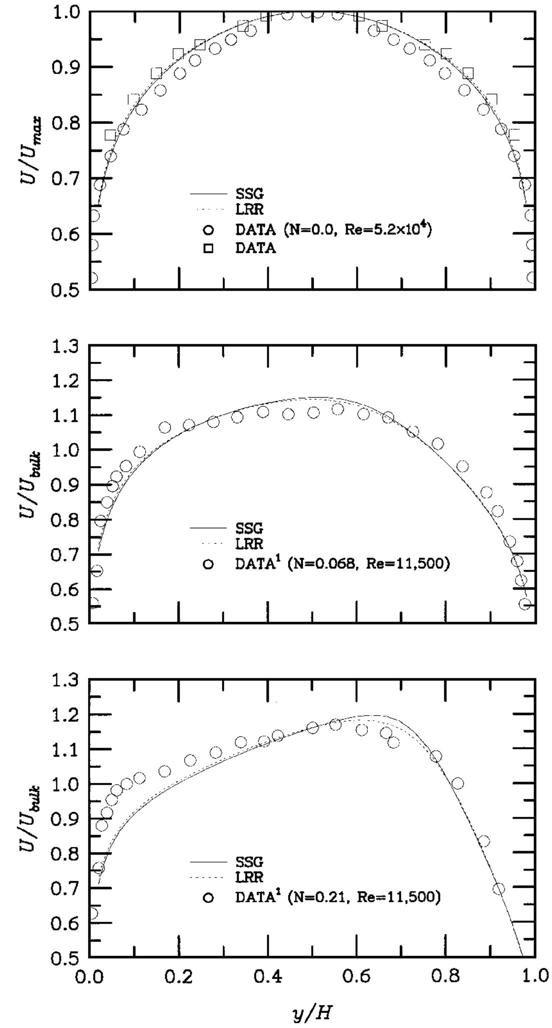

Fig. 1 Effects of rotation on channel mean velocity.

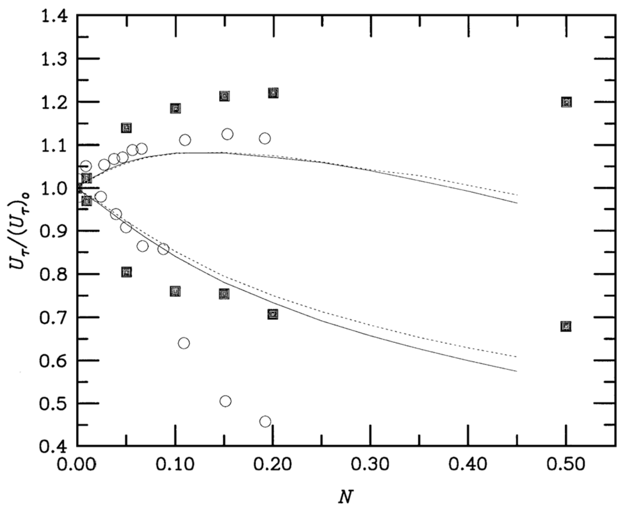

Fig. 2 Variation of friction velocity with rotation: ——, predictions ($Re = 3.5 \times 10^4$); - - - -, predictions ($Re = 1.15 \times 10^4$); ■, DNS (Ref. 5); and ○, data (Ref. 1).

fairly rapidly with N, reaches what appears to be a saturated state before actually beginning to fall with increasing N. At values of $N \geq 0.4$, the wall shear stress is reduced to levels below that of the stationary channel. The literature does not contain a reference to this unexpected behavior at high N, but this is not surprising because all of the previous calculations seem to be confined to $N \leq 0.15$.

Inspection of the shear- and rotation-production terms in the equation for $\overline{uv}$ provides a possible explanation for this behavior. These terms appear in that equation as

$$U_k \frac{\partial \overline{uv}}{\partial x_k} = -\overline{v^2} \frac{\partial U}{\partial y} - 2(\overline{u^2} - \overline{v^2})\Omega + \cdots \tag{6}$$

The quantity $(\overline{u^2} - \overline{v^2})$, although everywhere positive in a stationary channel, changes sign at high rates of rotation. Once this occurs, the rotation-production term becomes a sink in the $\overline{uv}$ equation, and this will act to reduce this quantity with increasing N. The DNS results support this interpretation: For $N = 0.5$, they show the difference in normal stresses to be negative over 63% of the channel cross section, with the levels of both the turbulence kinetic energy k and the structural parameter $(\overline{uv}/k)$ actually falling below their levels at $N = 0.15$. The causes of the exaggerated model behavior are not clear: It is possible that the pressure-strain models are not responding properly to the effects of high rotation or that the contribution of the pressure diffusion term can no longer be neglected in such conditions. Clearly, further work is needed to clarify this unexpected behavior.

Concluding Remarks

The applicability of linear and quadratic models for the fluctuating pressure-strain correlations is extended to flows with a system rotation by using the intrinsic spin tensor in place of the coordinate-dependent vorticity tensor. The extension is tested here for flow in a channel rotated about its spanwise axis but is equally applicable to all other models of rotation.

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Vortical Layer Analysis on Perturbed Conical Flow

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Nomenclature

a	= speed of sound
c_v	= specific heat at constant volume
G_{mn}	= shock-perturbation factor
K_δ	= hypersonic similarity parameter, $M_\infty \delta$
M_∞	= freestream Mach number
P	= pressure
S	= specific entropy
W	= azimuthal velocity perturbation
β	= semivertex angle of the unperturbed shock
γ	= ratio of specific heat (1.4 for air)
δ	= semivertex angle of the unperturbed cone
ε	= perturbation parameter (extremely small)
ρ	= density
σ	= shock relation, β/δ

Subscripts

m	= mth term in expansion
n	= nth term in expansion
0	= zeroth-order, unperturbed flow quantities

Introduction

REGARDING the development of the vortical layer problem, Ferri[1] proposed the vortical layer concept and analytically verified the invalidity of Stone's[2] results in an extremely thin vortical layer near the cone surface. This extremely small layer is required to adjust the entropy value from the outer solution to equalize the entropy value on the cone surface. Munsun[3] used the asymptotic matching principle and obtained uniformly valid solutions for the entire flowfield. He also found the azimuthal velocity and pressure in the outer expansion uniformly valid in the first-order perturbation. In 1985, Rasmussen[4] analyzed the shock layer in conical flow with transverse curvature. For inner expansion, he applied Munson's suggestion[3] of coordinate parameters and successfully obtained the analytical results for the vortical layer in the perturbed flow. Also, by using the matching method, the outer solutions and inner solutions are completely matched as the composite solutions, which are uniformly valid in the whole shock layer.

Recently, Lin and Rasmussen[5] combined the effects of longitudinal and transverse curvatures of conical flow and proceeded to carry out an outer perturbation expansion. Because the outer expansion solutions are not uniformly valid in the shock layer, however, the outcome near the conical body surface remains defective. To investigate the vortical layer near the cone surface for a conical body with multidirectional curvature, this study intends to discover uniformly

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